(1) $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ to plane polar

$x = r\cos\phi$

$y = r\sin\phi$

also $r = (x^2+y^2)^{1/2}$ $\phi = \tan^{-1}(y/x)$

chain rule:

$$\left(\frac{\partial f}{\partial x}\right)_y = \left(\frac{\partial f}{\partial r}\right)_\phi \left(\frac{\partial r}{\partial x}\right)_y + \left(\frac{\partial f}{\partial \phi}\right)_r \left(\frac{\partial \phi}{\partial x}\right)_y$$

$$\left(\frac{\partial f}{\partial y}\right)_x = \left(\frac{\partial f}{\partial r}\right)_\phi \left(\frac{\partial r}{\partial y}\right)_x + \left(\frac{\partial f}{\partial \phi}\right)_r \left(\frac{\partial \phi}{\partial y}\right)_x$$

evaluate the coefficients:

$$\left(\frac{\partial r}{\partial x}\right)_y = \frac{1}{2}(x^2+y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} = \frac{r\cos\phi}{r}$$

$$= \cos\phi$$

$$\left(\frac{\partial r}{\partial y}\right)_x = \frac{1}{2}(x^2+y^2)^{-1/2}(2y) = \frac{y}{r} = \frac{r\sin\phi}{r} = \sin\phi$$

$$\left(\frac{\partial \phi}{\partial x}\right)_y = \frac{1}{1+(y/x)^2}\cdot(-yx^{-2}) = \frac{-y}{(1+y^2/x^2)x^2} = \frac{-y}{x^2+y^2}$$

$$= \frac{-y}{r^2} = \frac{-r\sin\phi}{r^2} = \frac{-\sin\phi}{r}$$

$$\left(\frac{\partial \phi}{\partial y}\right)_x = \frac{1}{1+(y/x)^2}\cdot\left(\frac{1}{x}\right) = \frac{1}{x+y^2/x} = \frac{x}{x^2+y^2} = \frac{r\cos\phi}{r^2}$$

$$= \frac{\cos\phi}{r}$$

Combining

$$\left(\frac{\partial f}{\partial x}\right) = \cos\phi\left(\frac{\partial f}{\partial r}\right) - \frac{\sin\phi}{r}\left(\frac{\partial f}{\partial \phi}\right)$$

$$\left(\frac{\partial f}{\partial y}\right) = \sin\phi\left(\frac{\partial f}{\partial r}\right) + \frac{\cos\phi}{r}\left(\frac{\partial f}{\partial \phi}\right)$$

let $g = \left(\frac{\partial f}{\partial x}\right)$ and $h = \left(\frac{\partial f}{\partial y}\right)$

then

(1) $$\left(\frac{\partial^2 f}{\partial x^2}\right) = \left(\frac{\partial g}{\partial x}\right) = \left(\frac{\partial g}{\partial r}\right)\left(\frac{\partial r}{\partial x}\right) + \left(\frac{\partial g}{\partial \phi}\right)\left(\frac{\partial \phi}{\partial x}\right)$$

(2) $$\left(\frac{\partial^2 f}{\partial y^2}\right) = \left(\frac{\partial h}{\partial y}\right) = \left(\frac{\partial h}{\partial r}\right)\left(\frac{\partial r}{\partial y}\right) + \left(\frac{\partial h}{\partial \phi}\right)\left(\frac{\partial \phi}{\partial y}\right)$$

(1) $$\left(\frac{\partial^2 f}{\partial x^2}\right) = \cos\phi\left[\cos\phi\left(\frac{\partial^2 f}{\partial r^2}\right) - \frac{\sin\phi}{r}\left(\frac{\partial^2 f}{\partial r\,\partial\phi}\right) + \frac{\sin\phi}{r^2}\left(\frac{\partial f}{\partial \phi}\right)\right]$$

$$- \left(\frac{\sin\phi}{r}\right)\left[\cos\phi\left(\frac{\partial^2 f}{\partial\phi\,\partial r}\right) - \sin\phi\left(\frac{\partial f}{\partial r}\right) - \frac{\sin\phi}{r}\left(\frac{\partial^2 f}{\partial \phi^2}\right) - \frac{\cos\phi}{r}\left(\frac{\partial f}{\partial \phi}\right)\right]$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right) = \cos^2\phi\left(\frac{\partial^2 f}{\partial r^2}\right) - \frac{\sin\phi\cos\phi}{r}\left(\frac{\partial^2 f}{\partial r\,\partial\phi}\right)$$

$$+ \frac{\sin\phi\cos\phi}{r^2}\left(\frac{\partial f}{\partial\phi}\right) - \frac{\sin\phi\cos\phi}{r}\left(\frac{\partial^2 f}{\partial\phi\,\partial r}\right)$$

$$+ \frac{\sin^2\phi}{r}\left(\frac{\partial f}{\partial r}\right) + \frac{\sin^2\phi}{r^2}\left(\frac{\partial^2 f}{\partial\phi^2}\right)$$

$$+ \frac{\sin\phi\cos\phi}{r^2}\left(\frac{\partial f}{\partial\phi}\right)$$

$$= \cos^2\phi\left(\frac{\partial^2 f}{\partial r^2}\right) - \frac{2\sin\phi\cos\phi}{r}\left(\frac{\partial^2 f}{\partial r\,\partial\phi}\right)$$

$$+ \frac{2\sin\phi\cos\phi}{r^2}\left(\frac{\partial f}{\partial\phi}\right) + \frac{\sin^2\phi}{r}\left(\frac{\partial f}{\partial r}\right) + \frac{\sin^2\phi}{r^2}\left(\frac{\partial^2 f}{\partial\phi^2}\right)$$

2 $$\left(\frac{\partial^2 f}{\partial y^2}\right) = \sin\phi\left[\sin\phi\left(\frac{\partial^2 f}{\partial r^2}\right) + \frac{\cos\phi}{r}\left(\frac{\partial^2 f}{\partial r\,\partial\phi}\right)\right.$$

$$\left. - \frac{\cos\phi}{r^2}\left(\frac{\partial f}{\partial\phi}\right)\right]$$

$$+ \frac{\cos\phi}{r}\left[\sin\phi\left(\frac{\partial^2 f}{\partial\phi\,\partial r}\right) + \cos\phi\left(\frac{\partial f}{\partial r}\right)\right.$$

$$\left. + \frac{\cos\phi}{r}\left(\frac{\partial^2 f}{\partial\phi^2}\right) - \frac{\sin\phi}{r}\left(\frac{\partial f}{\partial\phi}\right)\right]$$

$$\left(\frac{\partial^2 f}{\partial y^2}\right) = \sin^2\phi\left(\frac{\partial^2 f}{\partial r^2}\right) + \frac{\sin\phi\cos\phi}{r}\left(\frac{\partial^2 f}{\partial r\,\partial\phi}\right)$$

$$- \frac{\sin\phi\cos\phi}{r^2}\left(\frac{\partial f}{\partial\phi}\right) + \frac{\sin\phi\cos\phi}{r}\left(\frac{\partial^2 f}{\partial\phi\,\partial r}\right)$$

$$+ \frac{\cos^2\phi}{r}\left(\frac{\partial f}{\partial r}\right) + \frac{\cos^2\phi}{r^2}\left(\frac{\partial^2 f}{\partial\phi^2}\right)$$

$$- \frac{\sin\phi\cos\phi}{r^2}\left(\frac{\partial f}{\partial\phi}\right)$$

$$= \sin^2\phi\left(\frac{\partial^2 f}{\partial r^2}\right) + \frac{2\sin\phi\cos\phi}{r}\left(\frac{\partial^2 f}{\partial r\,\partial\phi}\right)$$

$$- \frac{2\sin\phi\cos\phi}{r^2}\left(\frac{\partial f}{\partial\phi}\right) + \frac{\cos^2\phi}{r}\left(\frac{\partial f}{\partial r}\right)$$

$$+ \frac{\cos^2\phi}{r^2}\left(\frac{\partial^2 f}{\partial\phi^2}\right)$$

Then $\nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} =$

$$\left(\sin^2\phi + \cos^2\phi\right)\left(\frac{\partial^2 f}{\partial r^2}\right) + \frac{\left(\sin^2\phi + \cos^2\phi\right)}{r}\left(\frac{\partial f}{\partial r}\right)$$

$$+ \frac{\left(\sin^2\phi + \cos^2\phi\right)}{r^2}\left(\frac{\partial^2 f}{\partial\phi^2}\right)$$

$$\therefore \nabla^2 = \left(\frac{\partial^2}{\partial r^2}\right) + \frac{1}{r}\left(\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \phi^2}\right)$$