Do $x^2\frac{d}{dx}$ and $x\frac{d^2}{dx^2}$ commute?

$$\left[x^2\frac{d}{dx},\ x\frac{d^2}{dx^2}\right]f = x^2\frac{d}{dx}x\frac{d^2}{dx^2}f - x\frac{d^2}{dx^2}x^2\frac{d}{dx}f$$

$$= x^2\frac{d}{dx}x f'' - x\frac{d^2}{dx^2}x^2 f'$$

$$= x^2\left[x f''' + f''\right] - x\frac{d}{dx}\left[x^2 f'' + 2x f'\right]$$

$$= x^3 f''' + x^2 f'' - x\left[x^2 f''' + 2x f'' + 2x f'' + 2f'\right]$$

$$\neq 0$$ No!

what about $x\frac{d}{dx}$ and $x^2\frac{d^2}{dx^2}$

$$\left[x\frac{d}{dx},\ x^2\frac{d^2}{dx^2}\right]f = x\frac{d}{dx}x^2\frac{d^2}{dx^2}f - x^2\frac{d^2}{dx^2}x\frac{d}{dx}f$$

$$= x\frac{d}{dx}x^2 f'' - x^2\frac{d^2}{dx^2}x f'$$

$$= x\left[x^2 f''' + 2x f''\right] - x^2\frac{d}{dx}\left[x f'' + f'\right]$$

$$= x^3 f''' + 2x^2 f'' - x^2\left[x f''' + f'' + f''\right]$$

$$= 0$$ yes!

(2) if A and B are hermitian

ⓐ what about AB ?

is $\langle \psi | AB | \phi \rangle = \langle AB\psi | \phi \rangle$?

$\downarrow$

$\langle A\psi | B | \phi \rangle$ since A is hermitian

$\downarrow$

$\langle BA\psi | \phi \rangle$ since B is hermitian.

$\downarrow$

$\neq \langle AB\psi | \phi \rangle$ unless $BA = AB$ or $[A,B] = 0$

ⓑ what about $\frac{1}{2}(AB + BA)$?

is $\frac{1}{2}\langle \psi | AB + BA | \phi \rangle = \frac{1}{2}\langle (AB+BA)\psi | \phi \rangle$?

$\downarrow$

$\frac{1}{2}\left[\langle \psi | AB | \phi \rangle + \langle \psi | BA | \phi \rangle\right]$ $\qquad$ $\frac{1}{2}\left[\langle AB\psi | \phi \rangle + \langle BA\psi | \phi \rangle\right]$

$\downarrow$ since A & B are hermitian $\qquad$ $\downarrow$ since A & B are hermitian

$$\frac{1}{2}\left[\langle A\psi | B | \phi \rangle + \langle B\psi | A | \phi \rangle\right] \overset{?}{=} \frac{1}{2}\left[\langle B\psi | A | \phi \rangle + \langle A\psi | B | \phi \rangle\right]$$

✓

Note: this resolves ambiguity of order issues

e.g., xp_x is not hermitian but $\frac{1}{2}(xp_x + p_x x)$ is.

(c) show $A+iB$ and $A-iB$ are not hermitian

$$\langle\psi|A+iB|\phi\rangle = \langle\psi|A|\phi\rangle + \langle\psi|iB|\phi\rangle$$
$$= \langle A\psi|\phi\rangle + \langle B\psi|i|\phi\rangle$$

but if $A+iB$ is hermitian, this must equal

$$\langle(A+iB)\psi|\phi\rangle = \langle A\psi|\phi\rangle + \langle iB\psi|\phi\rangle$$
$$= \langle A\psi|\phi\rangle - i\langle B\psi|\phi\rangle$$

$\therefore$ not hermitian

$$\langle\psi|A-iB|\phi\rangle = \langle\psi|A|\phi\rangle - \langle\psi|iB|\phi\rangle$$
$$= \langle A\psi|\phi\rangle - i\langle B\psi|\phi\rangle$$

but $\langle(A-iB)\psi|\phi\rangle = \langle A\psi|\phi\rangle - \langle iB\psi|\phi\rangle$
$$= \langle A\psi|\phi\rangle + i\langle B\psi|\phi\rangle$$

$\therefore$ not hermitian

(3)

(a) Show that $2p_x$ and $2p_1$ are <u>not</u> orthogonal

$$\langle 2p_1 | 2p_x \rangle = \frac{1}{\sqrt{2}} \langle 2p_1 | 2p_{-1} + 2p_1 \rangle$$

$$= \frac{1}{\sqrt{2}} \left[\underbrace{\langle 2p_1 | 2p_{-1} \rangle}_{0} + \underbrace{\langle 2p_1 | 2p_1 \rangle}_{1} \right]$$

$$= \frac{1}{\sqrt{2}}$$ <u>not</u> orthogonal

(b) Schmidt orthogonalization:

let $|\phi_1\rangle = |2p_x\rangle$

then $|\phi_2\rangle = N_2 \left[|2p_1\rangle - |2p_x\rangle \langle 2p_x | 2p_1 \rangle \right]$

$$= N_2 \left[|2p_1\rangle - \frac{1}{\sqrt{2}} |2p_x\rangle \right]$$ using the result from part (a)

normalize

$$\langle \phi_2 | \phi_2 \rangle = 1$$

$$N_2^2 \langle 2p_1 - \tfrac{1}{\sqrt{2}} 2p_x | 2p_1 - \tfrac{1}{\sqrt{2}} 2p_x \rangle = 1$$

$$N_2^2 \left[\langle 2p_1 | 2p_1 \rangle - \tfrac{1}{\sqrt{2}} \langle 2p_1 | 2p_x \rangle - \tfrac{1}{\sqrt{2}} \langle 2p_x | 2p_1 \rangle + \tfrac{1}{2} \langle 2p_x | 2p_x \rangle \right] = 1$$

$$N_2^2 \left[1 - \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{2} \right] = 1$$

$$N_2^2 = 2 \quad ; \quad N_2 = \sqrt{2}$$

$$|\phi_2\rangle = \sqrt{2} |2p_1\rangle - |2p_x\rangle$$

Consider the operators for energy & angular momentum for the electron constrained to a ring of constant potential

$$-\frac{1}{2}\frac{d^2}{d\phi^2} \quad \text{and} \quad -i\frac{d}{d\phi}$$

a) Do these have a set of simultaneous eigenfunctions?

They do if the operators commute

$$\left[-\frac{1}{2}\frac{d^2}{d\phi^2}, -i\frac{d}{d\phi}\right]f = \frac{1}{2}i\left(\frac{d^2}{d\phi^2}\frac{d}{d\phi}f - \frac{d}{d\phi}\frac{d^2}{d\phi^2}f\right)$$

$$= \frac{1}{2}i\left(f''' - f'''\right) = 0 \qquad \underline{\text{yes!}}$$

b) Are there eigenfunctions for one but not the other?

remember that the eigenfunctions are $\psi = \frac{1}{\sqrt{2\pi}}e^{ik\phi}$

where $k = 0, \pm 1, \pm 2$, etc.

(also an eigenfunction of ang. mom.

so for $k \neq 0$, the states are 2-fold degenerate

e.g., for $k = \pm 1$ $\quad \psi_+ = \frac{1}{\sqrt{2\pi}}e^{i\phi} \quad \psi_- = \frac{1}{\sqrt{2\pi}}e^{-i\phi}$

or $\psi_+ = \frac{1}{\sqrt{2\pi}}\left[\cos\phi + i\sin\phi\right]$ and $\psi_- = \frac{1}{\sqrt{2\pi}}\left[\cos\phi - i\sin\phi\right]$

since these are degenerate, we are free to take linear combinations w/o affecting the eigenvalue, e.g.

$$\left.\begin{aligned} \psi_1 &= \tfrac{1}{\sqrt{2}}\psi_+ + \tfrac{1}{\sqrt{2}}\psi_- = \tfrac{1}{\sqrt{\pi}}\cos\phi \\ \psi_2 &= i\tfrac{1}{\sqrt{2}}\psi_+ - i\tfrac{1}{\sqrt{2}}\psi_- = \tfrac{1}{\sqrt{\pi}}\sin\phi \end{aligned}\right\} \text{ these are not eigenfunctions of ang. mom}$$