④ Show that $l_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$ in cartesians equals

$+i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)$ in

spherical polar coordinates

$$\boxed{\begin{array}{ll} y = r\sin\theta\sin\phi & r^2 = x^2+y^2+z^2 \\ z = r\cos\theta & \theta = \cos^{-1}(z/r) \\ x = r\sin\theta\cos\phi & \phi = \tan^{-1}(y/x) \end{array}}$$

$$\frac{\partial}{\partial z} = \left(\frac{\partial r}{\partial z}\right)\frac{\partial}{\partial r} + \left(\frac{\partial\theta}{\partial z}\right)\frac{\partial}{\partial\theta} + \left(\frac{\partial\phi}{\partial z}\right)\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y} = \left(\frac{\partial r}{\partial y}\right)\frac{\partial}{\partial r} + \left(\frac{\partial\theta}{\partial y}\right)\frac{\partial}{\partial\theta} + \left(\frac{\partial\phi}{\partial y}\right)\frac{\partial}{\partial\phi}$$

the coefficients:

$$\frac{\partial r}{\partial z} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z) = \frac{z}{r} = \cos\theta$$

$$\frac{\partial\theta}{\partial z} = \frac{-r}{\sqrt{r^2-z^2}}\quad\frac{r^2-z^2}{r^3}$$

$$= -\frac{\sqrt{x^2+y^2}}{r^2} = \frac{-r\sin\theta}{r^2} = \frac{-\sin\theta}{r}$$

$$\frac{\partial\phi}{\partial z} = 0$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta\sin\phi$$

$$\frac{\partial\theta}{\partial y} = \frac{-1}{\sqrt{1 - z^2/r^2}} \cdot -\frac{1}{2} z \left(x^2+y^2+z^2\right)^{-3/2} 2y$$

$$= \frac{r}{\sqrt{r^2 - z^2}} \cdot \frac{yz}{r^3}$$

$$= \frac{r}{r\sin\theta} \cdot \frac{r^2 \cos\theta \sin\theta \sin\phi}{r^3}$$

$$= \frac{\cos\theta \sin\phi}{r}$$

$$\frac{\partial\phi}{\partial y} = \frac{1}{1 + y^2/x^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2} = \frac{r\sin\theta\cos\phi}{r^2\sin^2\theta}$$

$$= \frac{\cos\phi}{r\sin\theta}$$

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so, $$l_x = -i\hbar \left\{ r\sin\theta\sin\phi \left[\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \right] - r\cos\theta \left[\sin\theta\sin\phi \frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \right] \right\}$$

$$l_x = -i\hbar \left\{ \sin^2\theta \sin\phi \frac{\partial}{\partial\theta} - \cos^2\theta \sin\phi \frac{\partial}{\partial\theta} - \frac{\cos\theta\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi} \right\}$$

$$= +i\hbar \left[\sin\phi \frac{\partial}{\partial\theta} + \cot\theta\cos\phi \frac{\partial}{\partial\phi} \right]$$