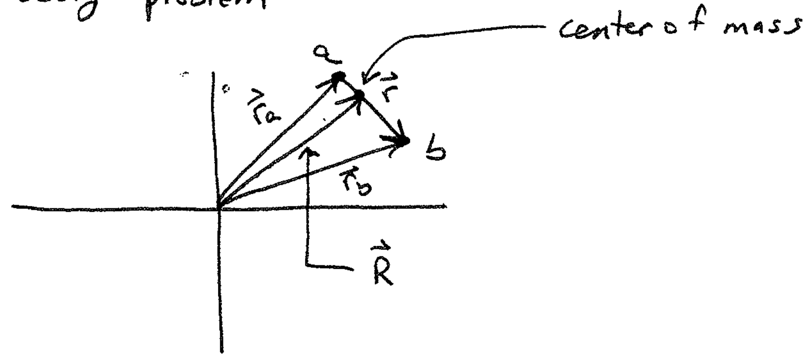
3 classical 2-body problem



$$H = \frac{1}{2}m_a\left(\dot{x}_a^2 + \dot{y}_a^2 + \dot{z}_a^2\right) + \frac{1}{2}m_b\left(\dot{x}_b^2 + \dot{y}_b^2 + \dot{z}_b^2\right) + V(r)$$

$$\vec{r}_a = (x_a, y_a, z_a) \qquad \vec{r}_b = (x_b, y_b, z_b)$$

$$\vec{R} = (X, Y, Z) \qquad \vec{r} = (x, y, z)$$

central relations: $\vec{R} = \dfrac{m_a\vec{r}_a + m_b\vec{r}_b}{m_a + m_b}$ $\qquad \vec{r} = \vec{r}_a - \vec{r}_b$

by substitution: $\boxed{\vec{r}_a = \vec{R} + \frac{m_b}{m}\vec{r} \qquad \vec{r}_b = \vec{R} - \frac{m_a}{m}\vec{r}}$

where $m = m_a + m_b$

in terms of the components, $x_a = X + \frac{m_b}{m}x$, etc

chain rule: $\dfrac{dx_a}{dt} = \underbrace{\dfrac{dx_a}{dX}}_{=1}\dfrac{dX}{dt} + \underbrace{\dfrac{dx_a}{dx}}_{\frac{m_b}{m}}\dfrac{dx}{dt}$

$$\dot{x}_a = \dot{X} + \frac{m_b}{m}\dot{x} \qquad \dot{x}_b = \dot{X} - \frac{m_a}{m}\dot{x}$$

etc.

Insert into H:

$$H = \frac{1}{2} m_a \left[\left(\dot{X} + \frac{m_b}{m} \dot{x} \right)^2 + \left(\dot{Y} + \frac{m_b}{m} \dot{y} \right)^2 + \left(\dot{Z} + \frac{m_b}{m} \dot{z} \right)^2 \right]$$

$$+ \frac{1}{2} m_b \left[\left(\dot{X} - \frac{m_a}{m} \dot{x} \right)^2 + \left(\dot{Y} - \frac{m_a}{m} \dot{y} \right)^2 + \left(\dot{Z} - \frac{m_a}{m} \dot{z} \right)^2 \right]$$

$$+ V(r)$$

expand:

$$H = \frac{1}{2} m_a \left[\dot{X}^2 + \frac{2 m_b}{m} \dot{X} \dot{x} + \left(\frac{m_b}{m} \right)^2 \dot{x}^2 + \dot{Y}^2 + \frac{2 m_b}{m} \dot{Y} \dot{y} + \left(\frac{m_b}{m} \right)^2 \dot{y}^2 \right.$$

$$\left. + \dot{Z}^2 + \frac{2 m_b}{m} \dot{Z} \dot{z} + \left(\frac{m_b}{m} \right)^2 \dot{z}^2 \right]$$

$$+ \frac{1}{2} m_b \left[\dot{X}^2 - \frac{2 m_a}{m} \dot{X} \dot{x} + \left(\frac{m_a}{m} \right)^2 \dot{x}^2 \right.$$

$$+ \dot{Y}^2 - \frac{2 m_a}{m} \dot{Y} \dot{y} + \dot{Z}^2 - \frac{2 m_a}{m} \dot{Z} \dot{z}$$

$$\left. + \left(\frac{m_a}{m} \right)^2 \dot{z}^2 \right] + V(r)$$

$$H = \frac{1}{2} (m_a + m_b) \left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 \right)$$

$$+ \frac{1}{2} \dot{x}^2 \left[m_a \left(\frac{m_b}{m} \right)^2 + m_b \left(\frac{m_a}{m} \right)^2 \right] + \frac{1}{2} \dot{y}^2 \left[m_a \left(\frac{m_b}{m} \right)^2 + m_b \left(\frac{m_a}{m} \right)^2 \right]$$

$$+ \frac{1}{2} \dot{z}^2 \underbrace{\left[m_a \left(\frac{m_b}{m} \right)^2 + m_b \left(\frac{m_a}{m} \right)^2 \right]}_{= \mu = \frac{m_a m_b}{m}} + V(r)$$

$$H = \frac{1}{2} m \left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 \right) + \frac{1}{2} \mu \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) + V(r)$$